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Some Identities and Inequalities for Derivatives

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We introduce some identities for the derivative of a trigonometric polynomial which are obtained from the identity of Riesz. We then use these new identities to derive some inequalities for derivatives of trigonometric and algebraic polynomials. Among our results are a weighted L^p inequality relating the derivative of a trigonometric polynomial to its L^p modulus and simple proofs for the inequalities of Brudnyi and Dzyadyk. We are able to give values to the constants in these inequalities. © 1995 Academic Press, Inc.

1. SOME OBSERVATIONS BASED ON THE IDENTITY OF RIESZ

The identity of Riesz [7] gives the derivatives of an arbitrary trigonometric polynomial Φ_m of degree at most m in the manner

$$\Phi'_m(z) = \frac{1}{4m} \sum_{k=1}^{2m} \Phi_m(z + t_k) \left((-1)^{k+1} \Big/ \left(\sin \frac{1}{2} t_k \right)^2 \right), \tag{1}$$

where $t_k := (2k-1)\pi/2m$. Setting $\Phi_m(z) = (1/m) \sin mz$ and $z = 0$ establishes that

$$1 = \frac{1}{4m^2} \sum_{k=1}^{2m} 1 \Big/ \left(\sin \frac{1}{2} t_k \right)^2, \tag{2}$$

a useful observation.

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Some new identities can be obtained from (1), by making specific choices of m and Φ_m. First of all, we may set $m := 2n$, and then

$$t_k := \frac{(2k-1)\pi}{2m} = \frac{(2k-1)\pi}{4n}.$$

Now we will assume that $\phi_n(z)$ is an arbitrary trigonometric polynomial of degree at most n and choose

$$\Phi_m(z) = \phi_n(z)\left(\frac{\sin(nz/2)}{n\sin(z/2)}\right)^2,$$

noting that $\Phi'_m(0) = \phi'_n(0)$. For $z = 0$ we thus obtain

$$\phi'_n(0) = \frac{1}{4m}\sum_{k=1}^{2m} \phi_n(t_k)\left(\frac{\sin(nt_k/2)}{n\sin(t_k/2)}\right)^2 \frac{(-1)^{k+1}}{(\sin(1/2)\,t_k)^2}.$$

Finally, since ϕ_n was arbitrary, we may set $\phi_n(z) := T_n(\theta + z)$, any polynomial of degree at most n, and we obtain

$$T'_n(\theta) = \frac{1}{4m}\sum_{k=1}^{2m} T_n(\theta + t_k)\left(\frac{\sin(nt_k/2)}{n\sin(t_k/2)}\right)^2 \frac{(-1)^{k+1}}{(\sin(1/2)\,t_k)^2}. \tag{3}$$

The derivation of (3) has been given in our recent contribution [3]. Clearly, it is also possible to choose $m = (r+2)\,n$ for $r = 0, \ldots$ and

$$t_k := \frac{(2k-1)\pi}{2m} = \frac{(2k-1)\pi}{2(r+2)\,n}. \tag{4}$$

Then one obtains

$$T'_n(\theta) = \frac{1}{4m}\sum_{k=1}^{2m} T_n(\theta + t_k)\left(\frac{\sin(nt_k/2)}{n\sin(t_k/2)}\right)^{2r+2} \frac{(-1)^{k+1}}{(\sin(1/2)\,t_k)^2}. \tag{5}$$

It is of course well known that from (1), (2), and the triangle inequality, the inequality of Bernstein follows immediately. Here, we show that the identities (3) and (5) can be used to prove some other inequalities involving derivatives.

2. An Inequality for Trigonometric Polynomials

First of all, (3) can be rewritten (using (3) for $T_n \equiv 1$) as

$$T'_n(\theta) = \frac{1}{4m}\sum_{k=1}^{2m} (T_n(\theta + t_k) - T_n(\theta))\left(\frac{\sin(nt_k/2)}{n\sin(t_k/2)}\right)^2 \frac{(-1)^{k+1}}{(\sin(1/2)\,t_k)^2}. \tag{6}$$

From (6) we obtain immediately

$$|T_n'(\theta)| \leqslant \frac{1}{4m} \sum_{k=1}^{2m} \frac{\omega(T_n; t_k) \sin^2(nt_k/2)}{(n \sin(t_k/2))^2} \cdot \frac{1}{(\sin(1/2)\, t_k)^2}.$$

Using (2) and recalling that $m = 2n$ in (6), we can conclude that

$$|T_n'(\theta)| \leqslant 2n \max_{k \leqslant m} \frac{\omega(T_n; t_k) \sin^2(nt_k/2)}{(n \sin(t_k/2))^2}.$$

in which we are free to take the maximum for $k \leqslant m$ instead of $2m$ by exploiting the periodicity of T_n. Using $\sin t < t$ for $t > 0$ and standard properties of the modulus of continuity, we obtain

$$|T_n'(\theta)| \leqslant 2n \max_{k \leqslant m} (2k-1)\, \omega\left(T_n; \frac{\pi}{4n}\right) \frac{\sin^2(nt_k/2)}{(n \sin(t_k/2))^2}.$$

Now for arbitrary indices k it is true that

$$\frac{\sin^2(nt_k/2)}{(n \sin(t_k/2)^2} \leqslant 1,$$

and for the indices $k \leqslant 2$ we can exploit this to establish

$$2n \max_{k=1,2} (2k-1)\, \omega\left(T_n; \frac{\pi}{4n}\right) \frac{\sin^2(nt_k/2)}{(n \sin(t_k/2))^2} \leqslant 6n\, \omega\left(\frac{\pi}{4n}\right).$$

For all $k \leqslant m$ we may of course use the fact that $\sin t \geqslant (2/\pi) t$ for $0 \leqslant t \leqslant \pi/2$, to obtain

$$2n(2k-1)\, \omega\left(T_n; \frac{\pi}{4n}\right) \frac{\sin^2(nt_k/2)}{(n \sin(t_k/2))^2} \leqslant \frac{32n}{2k-1}\, \omega\left(T_n; \frac{\pi}{4n}\right) \sin^2(nt_k/2),$$

which leads to a less sharp estimate for $k = 1, 2$. However, we may estimate $\sin^2(nt_3/2) < 7/8$ and $\sin^2(nt_4/2) < 1/2$ and in general $\sin^2(nt_k/2) \leqslant 1$ to obtain

$$2n \max_{k \leqslant m} (2k-1)\, \omega\left(T_n; \frac{\pi}{4n}\right) \frac{\sin^2(nt_k/2)}{(n \sin(t_k/2))^2} \leqslant 6n\, \omega\left(\frac{\pi}{4n}\right).$$

Therefore we have shown that

$$|T_n'(\theta)| \leqslant 6n\, \omega\left(T_n; \frac{\pi}{4n}\right). \tag{7}$$

The constant in the inequality (7) is good but not sharp; Stechkin [8] has obtained the similar inequality

$$\|T_n'\|_\infty \leqslant \left(\frac{n}{2\sin(nh/2)}\right) \|\Delta_h(T_n)\|_\infty,$$

in which $\Delta_h(f, t) = f(t+h) - f(t)$, and the choice of $h = \pi/4n$ gives a better estimate than (7). However, the argument by which (7) has been reached also applies in a much wider context. Let μ represent an arbitrary Borel measure defined on the interval $[0, 2\pi]$. Defining for $1 \leqslant p < \infty$ the norm

$$\|f\|_{p,\mu} = \left(\int_0^{2\pi} |f(\theta)|^p \, d\mu\right)^{1/p}$$

and defining a modulus

$$\omega_{p,\mu}(f; h) := \sup_{|s-t| \leqslant h} \|f(\theta+s) - f(\theta+t)\|_{p,\mu},$$

we obtain by exactly the same argument the following

THEOREM 1. *Let T_n denote an arbitrary trigonometric polynomial of degree at most n, and let μ be an arbitrary Borel measure. Then for $1 \leqslant p < \infty$ we have*

$$\|T_n'\|_{p,\mu} \leqslant 6n\, \omega_{p,\mu}\left(T_n; \frac{\pi}{4n}\right). \tag{8}$$

3. THE INEQUALITY OF BRUDNYI

Brudnyi's inequality gives a "local" estimate for the magnitude of derivatives of an algebraic polynomial $P_n(x)$, in the following manner:

THEOREM (Brudnyi [4]). *Let $P_n(x)$ be an algebraic polynomial of degree at most n which satisfies on $[-1, 1]$ the inequality*

$$|P_n(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^q \omega\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \tag{9}$$

in which q is a nonnegative integer and ω is a modulus of continuity. Then there is an absolute constant C_q such that

$$|P_n'(x)| \leqslant C_q \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-1} \omega\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right). \tag{10}$$

We will prove this inequality using (5) and give explicit estimates for the constants C_q.

Proof. We identify $x \in [-1, 1]$ with $\cos\theta$ for $\theta \in [0, \pi]$, obtaining $T_n(\theta) := P_n(\cos\theta)$. Then $P_n'(x) = T_n'(\theta)/-\sin\theta$, whence using (5) and $m = (r+2)\,n$ and the points t_k defined by (4)

$$P_n'(\cos\theta) = \frac{-1}{4m\sin\theta} \sum_{k=1}^{2m} P_n(\cos(\theta + t_k)) \left(\frac{\sin(nt_k/2)}{n\sin(t_k/2)}\right)^{2r+2} \frac{(-1)^{k+1}}{(\sin(1/2)\,t_k)^2}. \tag{11}$$

Therefore

$$|P_n'(\cos\theta)| \leqslant \frac{1}{4m\sin\theta} \times \sum_{k=1}^{2m} \frac{((|\sin(\theta+t_k)|/n) + (1/n^2))^q\,\omega((|\sin(\theta+t_k)|/n) + (1/n^2))}{(n\sin(t_k/2))^{2r+2}} \times \frac{\sin^{2r+2}(nt_k/2)}{(\sin(1/2)\,t_k)^2}.$$

Using (2) it is clear that

$$|P_n'(\cos\theta)| \leqslant \frac{m}{\sin\theta} \times \max_{k \leqslant 2m} \frac{((|\sin(\theta+t_k)|/n) + (1/n^2))^q\,\omega((|\sin(\theta+t_k)|/n) + (1/n^2))}{(n\sin(t_k/2))^{2r+2}} \times \left(\sin^{2r+2}\frac{nt_k}{2}\right). \tag{12}$$

To estimate the indicated maximum, we begin by assuming that $\sin\theta \geqslant (1/n)$. Since $m = (r+2)\,n$, it is advantageous to choose r as small as possible.

Beginning with the case $q = 0$, we should choose $r = 0$. We obtain from (12) that

$$|P_n'(\cos\theta)| \leqslant \frac{2n}{\sin\theta} \max_k \left(1 + \frac{|\sin(\theta+t_k)| + (1/n)}{\sin\theta + (1/n)}\right) \times \frac{\sin^2(nt_k/2)}{n^2\sin^2(t_k/2)}\,\omega\left(\frac{|\sin\theta|}{n} + \frac{1}{n^2}\right).$$

Using now the fact that

$$|\sin(\theta + t_k)| \leqslant |\sin \theta| + \left|2 \sin \frac{t_k}{2}\right|$$

we can obtain for $\sin \theta/(1/n)$ the estimate

$$|P'_n(\cos \theta)| \leqslant \frac{6n}{\sin \theta}\, \omega\left(\frac{\sin \theta}{n} + \frac{1}{n^2}\right)$$

from which

$$|P'_n(\cos \theta)| \leqslant 12\left(\frac{\sin \theta}{n} + \frac{1}{n^2}\right)^{-1} \omega\left(\frac{\sin \theta}{n} + \frac{1}{n^2}\right) \qquad \text{for} \quad \sin \theta \geqslant \frac{1}{n}. \qquad (13)$$

In order to obtain a similar estimate in the case that $q = 0$ but when $\sin \theta \leqslant (1/n)$, it is useful to obtain first a uniform estimate for $P'_n(\cos \theta)$ which is valid for $\sin \theta \geqslant (1/n)$. Beginning from (12) we obtain

$$|P'_n(\cos \theta)| \leqslant \frac{2n}{\sin \theta} \max_k \left(1 + \frac{|\sin(\theta + t_k)| + (1/n)}{(1/n)}\right) \frac{\sin^2(nt_k/2)}{n^2 \sin^2(t_k/2)}\, \omega\left(\frac{1}{n^2}\right),$$

and we obtain from this

$$|P'_n(\cos \theta)| \leqslant 10n^2\, \omega\left(\frac{1}{n^2}\right) \qquad \text{provided} \quad \sin \theta \geqslant \frac{1}{n} \qquad (14)$$

Now there is a uniform constant M such that, if p_n is a polynomial of degree at most n, then

$$\|p_n\|_{[-1,1]} \leqslant M \|p_n\|_{[-1+(1/n^2),\, 1-(1/n^2)]}.$$

More exactly (cf. Timan [10], 2.9(9) and 4.8(39)), it follows in our present context from (14) that

$$\|P'_n\|_{[-1,1]} \leqslant 10n^2 \left(\frac{n^2}{n^2 - 1}\right) \omega\left(\frac{1}{n^2}\right). \qquad (15)$$

It follows now from (15) combined with (13) that on the whole interval $[-1, 1]$ the inequality

$$|P'_n(x)| \leqslant 20\left(\frac{n^2}{n^2 - 1}\right)\left(\frac{\sqrt{1 - x^2}}{n} + \frac{1}{n^2}\right)^{-1} \omega\left(\frac{\sqrt{1 - x^2}}{n} + \frac{1}{n^2}\right) \qquad (16)$$

is satisfied, and we have demonstrated the inequality of Brudnyi for the case $q = 0$.

We move now to the case $q>0$, establishing the inequality first for the case that $0 \geqslant (1/n)$. It is advantageous to set $r := [q/2]$ in this case, and, inspecting the inequality (12) and referring to the argument leading up to (13), it is clear that we can obtain the estimate

$$|P_n'(\cos\theta)| \leqslant \frac{3(r+2)\,n}{\sin\theta} \times \max_k \frac{((|\sin(\theta+t_k)|/n)+(1/n^2))^q\,\omega((\sin\theta/n)+(1/n^2))}{(n\sin(t_k/2))^{2r+2}} \times \left(\sin^{2r+2}\frac{nt_k}{2}\right). \tag{17}$$

Observing that $2r=q$ if q is even and $2r+1=q$ if q is odd, we need now to estimate

$$\frac{3(r+2)\,n}{\sin\theta}\left(\frac{\sin\theta}{n}+\frac{1}{n^2}\right)^q \times \max_k \left(\frac{(|\sin(\theta+t_k)|/n)+(1/n^2)}{(\sin\theta/n)+(1/n^2)}\right)^q \frac{|\sin^q(nt_k/2)|}{|n\sin(t_k/2)|^q}.$$

Clearly, this expression is bounded for $\sin\theta \geqslant (1/n)$ by

$$6(r+2)\left(\frac{\sin\theta}{n}+\frac{1}{n^2}\right)^{q-1} \times \max_k \left(\frac{|\sin\theta\cos t_k|+|\cos\theta\sin t_k|+(1/n)}{\sin\theta+(1/n)}\right)^q \frac{|\sin^q(nt_k/2)|}{|n\sin(t_k/2)|^q},$$

which in turn is bounded by

$$6(r+2)\left(\frac{\sin\theta}{n}+\frac{1}{n^2}\right)^{q-1} \max_k \left(1+n\sin\frac{t_k}{2}\right)^q \frac{|\sin^q(nt_k/2)|}{|n\sin(t_k/2)|^q},$$

and we arrive at the estimate

$$|P_n'(x)| \leqslant 6\cdot 2^q\cdot\left(\left[\frac{q}{2}\right]+2\right)\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-1} \times \omega\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right) \quad \text{for } \sqrt{1-x^2} \geqslant \frac{1}{n}. \tag{18}$$

Several methods are available (with some sacrifice to the constant) for extending the estimate (18) from the subinterval $[-1+(1/n^2), 1-(1/n^2)]$

to the entire interval $[-1, 1]$. Perhaps the simplest of these to carry out is to use (5) twice, representing the second derivative of a trigonometric polynomial as a double sum. Then the derivative of an algebraic polynomial $P_n(x)$ can be effectively estimated for x near ± 1 by the second derivative of the related trigonometric polynomial $T_n(\theta)$. It should be clear, however, that this method will lead to a constant approximately equal to the square of the constant in (18). A second method is to obtain a uniform estimate of the form (15) for the qth derivative of P_n, then to integrate repeatedly until an estimate for P_n' is reached. This method is much more complicated than the first and is also numerically quite ineffective, since the cost incurred by taking q derivatives cannot be recovered.

A variant of the method followed in Timan [10], p. 221–222 seems to be more efficient than all of these. First, one must obtain an estimate in the spirit of (14), the proof of which is essentially a repetition of previous steps:

$$|P_n'(x)| \leqslant 5 \cdot 2^{q+1} \cdot \left(\left[\frac{q}{2}\right] + 2\right)\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-1} \times \omega\left(\frac{1}{n^2}\right) \quad \text{for } \sqrt{1-x^2} \geqslant \frac{1}{n}. \tag{19}$$

The domain of validity of this inequality includes the interval $[-1+(1/n^2), 1-(1/n^2)]$.

Now one defines a complex function, analytic outside of the interval $[-1+(1/n^2), 1-(1/n^2)]$, continuous on the interval, and attaining its maximum modulus on the interval. The function is

$$\frac{P_n'(z)}{[W_n(z)]^n \left(\sqrt{1-(1/n^2)-z^2}+(1/n)\right)^{q-1}},$$

in which

$$W_n(z) := \frac{n^2}{n^2-1}\left[z + \sqrt{z^2 - \left(1-\frac{1}{n^2}\right)^2}\right].$$

Noting that

$$|W_n(z)| = 1$$

for z real and lying in the closed interval $[-1+(1/n^2), 1-(1/n^2)]$, and that

$$\left|\sqrt{1-\frac{1}{n^2}-x^2}+\frac{1}{n}\right| \leqslant \sqrt{2}\left(\sqrt{1-x^2}+\frac{1}{n}\right) \quad \text{for } 1-\frac{1}{n^2} \leqslant |x| \leqslant 1$$

we obtain immediately for $\sqrt{1-x^2} \leqslant (1/n)$ the estimate

$$|P_n'(x)| \leqslant 2^{(q-1)/2}\left(\sqrt{1-x^2}+\frac{1}{n}\right)^{q-1} |W_n(x)|^n \times \left\| \frac{P_n'(z)}{[W_n(z)]^n (\sqrt{1-(1/n^2)-z^2}+(1/n))^{q-1}} \right\|, \tag{20}$$

in which the norm signifies the maximum modulus of the function. Since this norm is attained on the interval $[-1+(1/n^2), 1-(1/n^2)]$, we will in the remainder of the proof assume that z lies in this interval. Because on this interval $|W_n(z)| = 1$, we can also omit $W_n(z)$ in obtaining the remaining estimates.

To complete the estimate in (20), we must first estimate the factor $|W_n(x)|^n$. This quantity is clearly maximized if $|x| = \pm 1$, in which case we obtain

$$|W_n(x)|^n \leqslant \left(\frac{n^2}{n^2-1}+\sqrt{\left(\frac{n^2}{n^2-1}\right)^2-1}\right)^n \leqslant e^{\sqrt{2}} \quad \text{for} \quad 1-\frac{1}{n^2} \leqslant |x| \leqslant 1.$$

Therefore, from (20) combined with (19) we can obtain

$$|P_n'(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-1} e^{\sqrt{2}} \cdot 5 \cdot 2^{(3q-1)/2}\left(\left[\frac{q}{2}\right]+2\right) \times \left\| \frac{\sqrt{1-z^2}+(1/n)}{\sqrt{1-(1/n^2)-z^2}+(1/n)} \right\|^{q-1} \omega\left(\frac{1}{n^2}\right) \quad \text{for} \quad \sqrt{1-x^2} \leqslant \frac{1}{n}. \tag{21}$$

It remains to notice that

$$\left\| \frac{\sqrt{1-z^2}+(1/n)}{\sqrt{1-(1/n^2)-z^2}+(1/n)} \right\| \leqslant \frac{1+\sqrt{2-(1/n^2)}}{1+\sqrt{1-(1/n^2)}} \leqslant 2,$$

and we can obtain from (21) the estimate

$$|P_n'(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-1} e^{\sqrt{2}} \cdot 5 \cdot 2^{(3q+1)/2}\left(\left[\frac{q}{2}\right]+2\right) \times \omega\left(\frac{1}{n^2}\right) \quad \text{for} \quad \sqrt{1-x^2} \leqslant \frac{1}{n}. \tag{22}$$

Combining (22) with (18), we obtain an estimate which must hold on the entire interval $[-1, 1]$. This completes the proof of Brudnyi's inequality.

4. THE INEQUALITY OF DZYADYK

The inequality of Dzyadyk may be stated as follows:

THEOREM (Dzyadyk). *Let $P_n(x)$ be an algebraic polynomial of degree at most n which satisfies on $[-1, 1]$ the inequality*

$$|P_n(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^q \tag{23}$$

in which q is a nonnegative integer. Then

$$|P_n'(x)| \leqslant C_q \cdot \left(\frac{n^2}{n^2-1}\right)\left(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right)^{q-1}. \tag{24}$$

Instead of a modulus, ω, one has here a constant. Otherwise, the proof is quite similar to that for Brudnyi's inequality. Indeed, the constants are smaller because one does not need to deal with ω. For example, the constant in the case $q=0$ in Dzyadyk's inequality comes directly from the Markov-Bernstein inequality. Rather than to go through details of the proof of Dzyadyk's inequality here, we will prove two slightly more specialized results which have obvious usefulness in estimating the error in simultaneous approximation of derivatives by linear projections with augmented Hermite interpolation ± 1. Such approximation techniques have been developed in Balázs and Kilgore [1] and [2], Kilgore and Prestin [5], and tested with good results in Tasche [9], but the constants which govern the error estimates are as yet only known to exist. Other results similar to what follows are to be found in Rahman [6].

THEOREM 2. *Let $P_n(x)$ be an algebraic polynomial of degree at most n which satisfies on $[-1, 1]$ the inequality*

$$|P_n(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n}\right)^q \tag{25}$$

in which q is a positive integer and

$$\frac{P_n(x)}{(\sqrt{1-x^2})^q} \rightarrow 0 \qquad \text{as} \quad x \rightarrow \pm 1. \tag{26}$$

Then

$$|P_n'(x)| \leqslant (q+1)\frac{n-q}{n} \cdot \left(\frac{\sqrt{1-x^2}}{n}\right)^{q-1}. \tag{27}$$

Proof. We replace x by $\cos\theta$. Then we define a trigonometric polynomial

$$T_n(\theta)=\frac{P_n(\cos\theta)}{\sin^q\theta}$$

of degree at most $n-q$. We then note that

$$T'_n(\theta)=\frac{P'_n(\cos\theta)}{\sin^{q-1}\theta}-q\,\frac{\cos\theta P_n(\cos\theta)}{\sin\theta\sin^q\theta}, \tag{28}$$

whence

$$\left|\frac{P'_n(\cos\theta)}{\sin^{q-1}\theta}\right|\leqslant|T'_n(\theta)|+q\left|\frac{\cos\theta P_n(\cos\theta)}{\sin\theta\sin^q\theta}\right|. \tag{29}$$

Now, we need to estimate the second term on the right in (29). We note first that

$$T_n(0)=T_n(\pi)=0.$$

Thus, for $0<\theta<(\pi/2)$ we use $0<\theta<\tan\theta$ and the theorem of Rolle to make the estimate

$$\left|\frac{\cos\theta\,P_n(\cos\theta)}{\sin\theta\sin^q\theta}\right|<\left|\frac{T_n(\theta)}{\theta}\right|=|T'_n(\theta_1)|, \tag{30}$$

in which $0<\theta_1<\theta$.

Now it follows from the definition of T_n that

$$\|T_n\|\leqslant\frac{1}{n^q},$$

and therefore we can conclude from (30) and the inequality of Bernstein that

$$\left|\frac{\cos\theta\,P_n(\cos\theta)}{\sin\theta\sin^q\theta}\right|<(n-q)\frac{1}{n^q}. \tag{31}$$

The estimate (31) in in fact valid on the entire interval $[0,\pi]$, since it can be obtained for $(\pi/2)<\theta<\pi$ by a similar sequence of steps.

From (29) and (31) it follows in turn that

$$|P_n'(\cos\theta)| \leqslant (q+1)\frac{n-q}{n}\cdot\left(\frac{\sin\theta}{n}\right)^{q-1},$$

and (27) is established. This ends the proof of Theorem 2.

THEOREM 3. *Let $P_n(x)$ be an algebraic polynomial of degree at most n which satisfies on $[-1, 1]$ the inequality* (25), *that*

$$|P_n(x)| \leqslant \left(\frac{\sqrt{1-x^2}}{n}\right)^q \tag{32}$$

in which q is a positive integer. Then

$$|P_n'(x)| \leqslant (q+1)\frac{n-q}{n}\cdot\left(\frac{\sqrt{1-x^2}}{n}\right)^{q-1} \quad \text{holds for odd } q \tag{33}$$

and

$$|P_n'(x)| \leqslant \frac{n-q}{n}\cdot\left(\frac{\sqrt{1-x^2}}{n}\right)^{q-1} + \frac{q|x|}{n^2}\cdot\left(\frac{\sqrt{1-x^2}}{n}\right)^{q-2} \quad \text{holds for even } q. \tag{34}$$

The constants in these two inequalities are best possible.

Proof. We first notice that, with $x = \cos\theta$ and q odd, the polynomial $n^{-q}\sin^q\theta\,\sin(n-q)\,\theta$ is extremal in (33), and with q even the polynomial $n^{-q}\sin^q\theta\,\cos(n-q)\,\theta$ is extremal in (34). These observations will show that the stated constants in both inequalities are best possible, when combined with the rest of the proof.

Theorem 3 follows immediately from Theorem 2 if q is odd, since in that case (26) follows automatically. On the other hand, if q is even the polynomial P_n satisfying (25) may not satisfy (26). To remedy this situation, we must first notice that, since P_n satisfies (25), we can still define $T_n(\theta)$ as in the proof of Theorem 2 by

$$T_n(\theta) = \frac{P_n(\cos\theta)}{\sin^q\theta}.$$

We note that as before T_n is a trigonometric polynomial of degree at most $n-q$ and satisfies

$$\|T_n\| \leqslant \frac{1}{n^q}.$$

Then we have

$$\sin \theta T_n(\theta) = \frac{P_n(\cos \theta)}{\sin^{q-1} \theta},$$

and therefore

$$-\frac{P_n'(\cos \theta)}{\sin^{q-2} \theta} - (q-1) \frac{\cos \theta \, P_n(\cos \theta)}{\sin^q \theta} = \sin \theta T_n'(\theta) + \cos \theta T_n(\theta).$$

But then

$$-P_n'(\cos \theta) = \sin^{q-1} \theta T_n'(\theta) + q \cos \theta T_n(\theta) \sin^{q-2} \theta,$$

and we obtain, using Bernstein's inequality,

$$|P_n'(\cos \theta)| \leqslant |\sin^{q-1} \theta| \cdot \frac{n-q}{n^q} + q \, |\cos \theta| \, \frac{|\sin^{q-2} \theta|}{n^q},$$

from which (34) follows immediately. This ends the Proof of Theorem 3.

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